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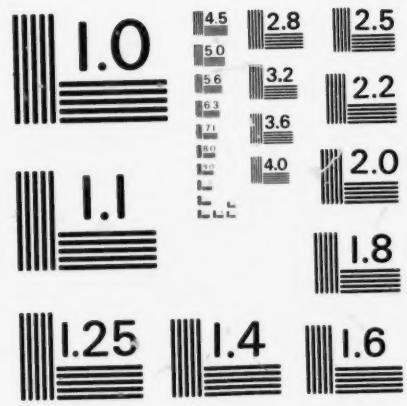

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DOCUMENT RESUME

ED 035 547 SE 007 726

AUTHOR	Swain, Henry
TITLE	How to Study Mathematics, A Handbook for High School Students.
INSTITUTION	National Council of Teachers of Mathematics, Inc., Washington, D.C.
PUB DATE	55
NOTE	32p.
AVAILABLE FROM	National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington, D.C. 20036
EDRS PRICE	EDRS Price MF-$0.25 HC Not Available from EDRS.
DESCRIPTORS	Homework, Learning, *Mathematics Education, *Secondary School Mathematics, Study Guides, *Study Skills

ABSTRACT

Study methods which will enable students to do well in high school mathematics are discussed in this booklet. Suggestions are offered concerning homework, classwork, taking tests, and special aids for studying particular areas of mathematics. Tips on doing homework include how to use the textbook, how to memorize in mathematics, how to avoid making careless errors, how to review, and how to use a notebook. Effective use of class time, note taking, and reviewing for and taking tests are mentioned. Special helps for studying algebra, geometry, and trigonometry are pointed out. This section presents ideas concerning such topics as: (1) Distinguishing equations from expressions, (2) Studying story problems, (3) Studying geometry theorems, (4) Organizing geometry definitions and axioms, (5) Thinking out original proofs, (6) Proving trigonometric identities, and (7) Mastering trigonometric formulas. (FL)

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HOW TO STUDY MATHEMATICS

a handbook for high-school students

HENRY SWAIN

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First Printing, February 1955
Second Printing, September 1955
Third Printing, November 1956
Fourth Printing, December 1959
Fifth Printing, December 1961
Sixth Printing, March 1965
Seventh Printing, November 1966
Eighth Printing, January 1969

PRINTED IN THE UNITED STATES OF AMERICA

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Introduction

Why aren't you getting better grades in mathematics? Do you feel that you have been putting in all the time on it that can be expected of you and that you are still not getting results? Or are you just lazy? If you *are* lazy, this handbook is not intended for you—unless from it you are fortunate enough to catch that eagerness to try hard. But if you have been trying and your grades still don't show your ability, or if you have been getting good grades but still feel that the mathematics does not mean very much to you, it is very likely that you do not know how to study *effectively*. This booklet aims to help you, its readers, to study mathematics effectively.

What does studying include? Studying is often thought of in the narrow sense of homework, but actually effective study must be thought of in a broader sense, including not only homework, but also classwork and tests—the whole of the learning process.

This pamphlet will not attempt to discuss the more general principles of good studying such as planning your study time; having a good place to study, with proper equipment, light, etc.; having no unnecessary distractions; getting started; having the will to learn. These points are treated thoroughly in general booklets on "How to Study." We are concerned here with a specific problem—how to study *mathematics*.

Some of you, especially if you are doing well in mathematics, may feel that you have successful study methods of your own different from the ones described here. In that case you need not feel you must change your methods, although you might profit from comparing your methods with these.

On the other hand, some of you may feel that the suggestions on the following pages are over-ambitious—that they would require more time and effort than you are prepared to give. You will probably be right. We cannot expect to do everything to perfection. But we can do the best we are able, and out of the suggestions offered, you can pick the ones that may help you most, and as you find your work improving, you may be able to try further suggestions. So scoff if you wish at these ambitious suggestions, but then give some of them a try, a *fair* try, and watch the results.

I

HOMEWORK

1. A four-step routine for doing your daily homework

There is a common misconception that homework is primarily something to hand in to the teacher. Actually, the homework is first and foremost a means of learning fundamental ideas and processes in mathematics, and of developing habits of neatness and accuracy. What is passed in to the teacher is only a by-product of that learning process. Home study to be effective should concern itself primarily with ideas and processes, not just with the completion of a written assignment. The following four-step routine is a suggestion for making your home study effective.

a. *Get oriented.* Take a few minutes to think back, look over your notes, and look over the book to see clearly what ideas you have been working on in your mathematics during the last week or so.

b. *Line up the ideas.* Think about the ideas, principles, and methods in today's lesson. Use the "study sheet" method described below to help you see them clearly. Don't forget to master any new words in your mathematics vocabulary.

c. *Write the assignment.* As you do the exercises assigned, think about the ideas they are illustrating. Remember that your goal is not just to get answers but rather, through practice, to increase your understanding and to develop your skill. (See Article 6, *Pointers on doing the written homework.*)

d. *Summarize.* Before you stop, while your mind is still "in the groove," summarize the ideas on which you have been working. Try saying them aloud as though you were explaining them to someone who hasn't learned them yet; there is no better way to learn a topic than by trying to teach it!

2. *How to use the "study sheet" method*

A good way to clarify your thinking as you study is to use a "study sheet." This is a sheet of paper on which to write down in your own words the ideas of the lesson—not exercises, but statements of "Why" and "How." Use an outline form to organize the different parts of the topic you are studying. Make lists of important words. State warnings about errors to avoid As you read material in your textbook, write down the main points of the assignment and work out the details of the examples to be sure you really understand the concepts they are illustrating. By thinking on paper you are more likely to master what you are studying, than when you just look at the book.

The main purpose of the "study sheet" is accomplished as you write it, and the study sheet method is worth doing even if you discard the sheets. But if you file these sheets away day by day, you will probably find them very helpful later on for review.

3. *How to use the textbook*

If you are using a textbook in your mathematics course, have you learned how to take advantage of all the help it can give you? Some of these suggestions may be useful.

a. Leaf through frequently and get an over-all view of what you are studying instead of just looking at one or two pages each day.

b. Look through the table of contents occasionally to see what you've been doing and what is to come.

c. Use the index, especially when you've forgotten the meaning of a word.

d. When your book gives an example to illustrate an idea, analyze the example carefully for the *ideas* behind it instead of just trying to make your exercises look like the example.

e. If you can't do an exercise, reread the explanatory material in the book slowly and carefully instead of giving up and waiting for someone to tell you how to do it.

f. Make the most of the study helps at the end of each chapter, such as lists of important words, outlines of the material in the chapter, review questions, and self-tests.

g. Keep an open mind about the explanatory material in your book. Most textbooks are not perfect, and you may find your teacher showing you a better viewpoint in some places. If so, be sure to get the full meaning of the new approach. If you are permitted to do so, you should make a note about it in the margin of the book. Similarly if you think you see a clearer way of explaining a topic or exercise, don't hesitate to discuss it with your teacher.

h. If an explanation in your textbook is not clear to you, try reading the corresponding explanation in one or two other mathematics books. Frequently studying two or three different presentations of the same idea makes the idea much clearer. Your teacher or your school library may be able to lend you the books you want.

4. How to develop your mathematics vocabulary

Suppose you said to me, "Att iakttaga vid spelets montering," I wouldn't be able to understand you because I don't know the Swedish language. Or suppose a radio ham tells you, "A class 'A' amplifier is biased to the midpoint of the tube's E_G,I_P curve," the chances are that you won't know what he is talking about because you don't know the technical language of radio. Similarly, without a knowledge of the vocabulary of mathematics you will not be able to follow what is being said in mathematics. Furthermore you will have difficulty in expressing yourself. If you have trouble saying clearly what you mean on tests, improving your vocabulary is the first thing you need to do. Here are some suggestions for developing your mathematics vocabulary:

a. Whenever you come to a new word in the book:

(1) Think carefully, "What does this word mean?"

(2) Analyze the word into its parts and associate it with other more familiar words.

(3) Practice spelling it (write it).

(4) Practice using it intelligently in some good sentences.

(Textbooks are usually quite helpful about italicizing new words when they are introduced and defined.)

b. If in your reading or in classwork you run across an old word, the meaning of which you have forgotten, look it up (use the index in your book, or a dictionary) and repeat the process in *a* above.

c. Make a conscious effort to *use* your mathematical vocabulary regularly and intelligently.

d. As a double check on yourself, keep a page in your notebook entitled "My Mathematics Vocabulary" where you list each new word in the subject as you come to it (just the word, not the definition). Then once a week (set a definite day and time in your schedule and stick to it) read through your list of words, thinking what each word means. Mark any word about which you are not sure, look it up, practice it, and next time be particularly careful to double check on that word. Cross out words as you become thoroughly familiar with them.

5. *How to memorize in mathematics*

Actually there is very little pure memory work in mathematics. Most of the work is a matter of reasoning and understanding. Some people may talk about memorizing definitions, postulates, and theorems in geometry. What really happens—or *should* happen—is that you learn to *understand* the ideas involved and associate them with a mental picture of the geometric situation. Then you can formulate the words yourself.

There are a few situations which involve pure memory. These include:

(1) Formulas

(2) Frequently used constants, such as π, $\sqrt{2}$, $\sqrt{3}$, etc.

(3) The number facts of arithmetic (addition and multiplication tables, etc.)

Some formulas follow so immediately from other formulas that they can easily be remembered by thinking of how they

are derived. You can quickly recall such formulas as the one for the area of a triangle (since a triangle is half of a parallelogram), the formula for the sum of an infinite geometric progression $S = \frac{a}{1 - r}$ $\left(\text{which comes from } S = \frac{a - ar^n}{1 - r}\right)$, or the formula for $\sin 2\theta$ (which comes from $\sin [x + y]$). In such cases, while you want to be able to use the derived formula directly, it is good to refresh your memory frequently by recalling the relationship with the earlier formula.

A few formulas and certain constants which are often needed require actual memorizing. If you have trouble learning them, there are various devices that can help you. For instance, write the constants or formulas on a card and clip the card to a book or notebook which you carry to and from school. Then as you ride your bus, or even as you walk, first study the card, then turn it over and see how accurately you can repeat the numbers or formulas. The next day try repeating them *before* you look at the card, and then check yourself. Keep at them until you really know them. This card method can also be used at the beginning of your homework or in the few minutes that are so often wasted between activities at home. Try various methods, and once you have learned your formulas or constants, keep using them so you don't forget what you have learned.

Most high-school students know the number facts of arithmetic, but once in a while someone slips through without having learned them. If you are one of those, realize that there *is* something you can do about it; you don't have to continue to suffer under that handicap. Probably flash cards are one of the best devices for learning your number facts. You can easily make such cards yourself, using the number combinations on which you need to practice. Get a parent or friend to help you go through them, naming results as rapidly as you can with accuracy. Keep at them until you can go through all the number facts perfectly.

6. *Pointers on doing the written homework*

Don't forget that the main objective of the written homework is to *learn*—to learn the ideas of mathematics by putting them into practice, and also to learn habits of neatness, accuracy, and clarity of expression.

The written homework sometimes has also the objective of *showing how much you have learned*. For both of these purposes of the homework the following pointers will help you do a better job.

a. *Get the assignment accurately*. Have a definite place in your notebook where you write down the assignment for mathematics each day. If you aren't sure what the teacher means by any part of the assignment, don't hesitate to ask about it and get it clarified.

b. *Follow directions*. Read the instructions at the beginning of the exercises and follow them.

c. *Work neatly and accurately*. The habits of neatness and accuracy which you develop in doing this homework will be of importance to you later.

d. *Show your complete work*. Show on your paper all work which is not done mentally. This will help you and your teacher when you are checking through for errors. It also develops a good habit for later work.

e. *Check the reasonableness of your answer*. Learn to make estimates to check the correctness of your work.

f. *Do the work promptly*. By doing the work promptly, you fix the principles in your mind before you have forgotten all the instructions.

7. What to do if you "get stuck" on your homework

Even the best of students will sometimes run across an exercise in the homework which he can't do. If you just give up and forget about it, you are very likely to lose out on an idea you will need. The first thing to do is to look back at the book, at your notes, or at your "study sheet" for ideas related to the problem on which you are working. If you have been thinking in terms of "How's" and "Why's," you will probably find the answer to your difficulty very soon yourself. If your work on a problem seems to be completely confused, it sometimes helps to discard your paper entirely and start afresh. If you still can't

clear up your thinking on the problem, make yourself a reminder and ask about it as soon as possible, perferably during the class discussion, but otherwise by arranging to ask the teacher outside of class. Your reminder can be a note to yourself clipped to your homework or to the appropriate page in your book. Whatever happens, be sure to follow through on getting the difficulty cleared up.

8. *What about getting help from your parents, or from other students?*

Many parents are frequently called on to help do mathematics homework, and many telephones ring every night for Sally to ask Suzy, "How do you do the ninth one in the algebra?" Is this right?

Yes, if it is done the right way. Obviously if Dad or Suzy just gives the answer, Sally doesn't learn very much. But if the help is based on a mutual discussion of the "How's" and "Why's" of the situation, if the conversation can be on the basis of not "What's the answer?" but "What's the point of this?" then it can be very wholesome. Bull sessions about the mathematics *after* you have done your best, are good experiences.

One difficulty you may encounter is that your dad may have been taught by a method which has now been replaced by a different one. Unless you can explain clearly to him the method your class is using, his help may be more confusing than otherwise.

Whether or not it is right to collaborate on written work depends on the nature of the work. A project which is going to be graded for individual credit, of course, should be done independently.

9. *How to avoid making careless errors*

There is no magic formula which will immediately eliminate all your careless errors. Perhaps the nearest thing is really to care. If you are seriously conscious of your weakness and sufficiently disturbed every time a foolish mistake slips in, you will be less likely to make those mistakes.

Some people have found that the "check-back" method helps. The idea is to check back every few seconds over what you have

just done to see if it is really what you meant, instead of rushing through to the end of a whole problem and then trying to go through and catch your mistakes. It may seem to take longer at first to check back but in the long run it will save you time by helping you catch your mistakes early. When checking back becomes a habit, you will be a much more accurate worker.

Another good thing to do is to look at your answer critically and see whether it is reasonable. If you get an answer of 248 miles per hour for the rate at which Mr. Bong walked from his house to his office, you can know that something is wrong. Whenever you find an unreasonable answer, go back over the problem and find the error.

10. How to make your errors help you learn

What do you do when an answer is wrong in your homework, or on a test? Do you throw it away and forget it—and then make the same mistake the next time? If you are wise, you'll make those errors teach you something. Here's what you can do:

a. Analyze the error to see what kind it is and what caused it.

b. If it is a careless error and you really knew how to do the work correctly, make a note of it and if you find that you keep making careless errors frequently, start putting on a campaign to overcome that habit of careless work. See the preceding article for suggestions on such a campaign.

c. If you can't see what caused the error, search your mind, the book and your notes for ideas that the exercise was intended to illustrate. See whether you applied those ideas correctly.

d. If you still can't find where your error is, ask a teacher to help you to see what is wrong.

e. Now that you have found what you did wrong, *do* something about it. Practice by doing that kind of exercise the *right* way at least twice.

f. Keep a page in your notebook entitled "Warning: Errors to Avoid." Write there a description of the right way to do that kind of exercise, being sure to emphasize the important idea behind it. For instance:

In simplifying fractions look for *factors* which are the same in the numerator and denominator and which will thus form a factor *one*. For example:

$$\frac{3a-6}{6a} = \frac{3(a-2)}{3 \cdot 2a} = \frac{3}{3} \cdot \frac{a-2}{2a} = \frac{a-2}{2a}$$

Here $\frac{3}{3}$ can be replaced by 1, since $\frac{x}{x} = 1$ for all numbers x, providing $x \neq 0$.

We know also that 1 is the *identity* number for multiplication, which means that for all numbers y, $y \cdot 1 = y$.

Remember, if these suggestions seem too ambitious for you, at least they give you something at which to aim.

11. How to review as you go along

It is very easy, if you aren't careful, to get into the habit of doing each day's assignment well enough, but then never thinking of it again, with the result that the ideas you are studying just pass through your mind and very few of them stay there. This seems to be the case with students who do well in their daily work, but do poorly on the major tests. What is needed is some sort of systematic regular review instead of trying to cram all the reviewing in just before the big tests. This is easier than you may think, and it "pays off" in the long run.

The thing to do is to set a time at least once a week when you take a few minutes to survey what you have been doing. This survey should consist of two parts:

(1) A careful study of what you have been doing for the last week, with an effort to emphasize the important ideas.

(2) A broader look at the whole preceding part of the course, seeing how the recent work is related to the rest and spotting any portions of the course about which you are not confident.

In both these parts most people can review more quickly and efficiently not by doing a lot of exercises, but by writing outlines of the ideas. For the more recent work, the outline should be in considerable detail, giving explicit directions on *how* to proceed and *why* we can and should proceed that way. If you do

practice on exercises, select them intelligently, thinking about how the ideas you have been studying can be applied.

For the broader survey of the earlier work the outline should be more general, giving a bird's-eye view of the whole course. If you feel uncertain about an earlier section of the work, get out the detailed outline which you made when you first reviewed that part, study it over, and try a few exercises to refresh your understanding.

If two or more students will get together and discuss their review outlines, the interaction of ideas will usually be particularly beneficial.

12. *How to use a notebook in mathematics*

Frequent mention has been made in these pages of a mathematics notebook. Intelligent use of a notebook can do a great deal toward making your mathematics easier and increasing your understanding. Here is a resumé of some of the sections you might have in your notebook:

a. An assigment page
b. A place for class notes (See Article 14.)
c. A place for your vocabulary list (See Article 4.)
d. A place for your "study sheets" (See Article 2.)
e. A place for warnings of errors to avoid (See Article 10:f.)
f. A place for your review outlines (See Article 11.)

II

CLASSWORK

13. How to make the most of your time in class

a. Get ready. In the minute or two before the class gets started, think over what you have been working on recently and have your mind "warmed up," ready to go.

b. Have all necessary equipment: book, sharpened pencils, notebook, ruler, compasses, tables, etc.

c. Take down the assignment promptly and accurately.

d. Concentrate. This takes an effort if you are the kind whose mind easily wanders off to other subjects. When you start thinking about that party you went to Saturday, or how pretty Mary is, pull your mind back sharply and remember that you need to take advantage of every minute in class or you will miss something.

e. Ask questions when you don't understand.

f. Listen to the questions and answers of others in the class. When another pupil is reciting, think how *you* would answer the question.

g. Take part in the discussion. Not only will it help the class to have you participate, but it will help you if you join in and express yourself.

h. Try hard to grasp the ideas of the lesson. Be sure to get the *whole idea*, not just part of it.

i. Do not write at the wrong time. If you take any notes or make any corrections on your homework paper, be sure you do not miss anything that is said while you are doing so. Most people cannot write and listen at the same time. (See article 14.)

j. As soon after class as you can, decide what were the main points of the class lesson.

14. How to take notes in class

Whether or not you take notes in your mathematics class will depend on how the class is run. If the teacher encourages you to take notes, have a place ready in your notebook to which you can turn quickly if there is something to take down. Sometimes the teacher will give you an outline or some special pointer which is important to remember and will tell you to take it down. Sometimes an idea will be developed in class which you yourself feel is worth writing down for future reference. In either case there are two conflicting things you must try to do. One is to make your notes complete and accurate enough to be valuable to you later. The other is to make your notes brief enough so that you can continue to listen to what is being said in class. It is an art to do both of these at once—an art which you will need to develop before you go to college.

One more thing—if you do take notes, be sure that you *use* them to good advantage later. Study them soon after class to fix the ideas in your mind. In particular, let them help you in your weekly review.

III

TESTS

15. How to review for tests

(Note: read this several days *before* the test.)

Here are some pointers that can help you do a good job of reviewing:

a. Start reviewing far enough in advance so you have time to do a careful unhurried job, and still are able to go to bed early the night before the examination. If you have been reviewing as you go along, this major review will not take long.

b. If you have some review outlines you have already made, get them out and from them make a master outline of the whole section of the course on which you are being examined.

c. Go over the parts of the master outline one at a time in considerable detail using your detailed outlines, your class notes, your textbook, your reminders of dangers to avoid, and perhaps your "study sheets" to make sure that you have a clear picture of the ideas in that part.

d. If you have none of these helps from your earlier work, you will have to make your outline right from scratch, using the textbook and your head as the only sources of the ideas—and it will be all the more important that you do so. Make your outline concise and to the point. Such an outline conscientiously worked out with the details carefully thought through can make a tremendous difference in the results of a final examination.

e. In some parts of this outline you will feel absolutely sure of yourself and it is probably unnecessary to do any more about them. Other parts may be less clear and for those it might be good to pick out some exercises and try doing them with the

ideas of your outline in mind. This application of the ideas to some particular examples will often clarify them in your mind.

f. Don't forget to review your mathematics vocabulary.

g. If there are some formulas for which you are responsible, make a list of them and then practice saying them, or writing them, using the list only as a check on yourself.

h. Many books have summaries, word lists, and review material at the end of each chapter. These are excellent if you use them intelligently, letting them be a guide to points you need to re-study more carefully.

i. If you were the teacher, what questions would you ask on the test? Prepare yourself for those questions.

j. Get a good night's rest the night before the examination.

k. *Don't worry.*

16. How to take tests

a. When you take a test, have the right attitude—take pride in doing the best job you can. Don't try to "get by" with doing as little as possible. Have confidence in your own ability, but don't be overconfident.

b. Be serious and concerned enough about the test to do your best, but don't worry to the point of anxiety. Fear alone can make a person do poorly on a test regardless of his ability and knowledge.

c. Have all necessary equipment such as sharpened pencils, eraser, ruler, and compasses.

d. Follow directions. Read carefully and listen carefully for any special instructions, such as where answers are to be written, how many questions must be answered, etc. Follow directions in each individual question, too.

e. Look over the whole test quickly at the start and, unless you are required to do the questions in the order given, do the ones you are sure of first.

f. If you are unable to answer a question, leave it and go on to others, coming back to the hard one later. Often with a fresh start you will suddenly see much better what to do.

g. Without fussing about the time, give an occasional glance at the clock so that you can make the most of the time you have. Many tests are planned so that a large number of the students will not finish all the questions. Don't be upset if the time is running out and you still have more to do. If you have done your best and have not wasted time, nothing more is expected of you.

h. Be careful to show clearly what you are doing. Put in enough steps so that your method is perfectly clear. Remember that the teacher is not a mind reader, and your grade may depend on whether or not the teacher can see from your work that you understand what you are doing.

i. Work neatly. To work neatly is a good habit in itself. And it makes a good impression on the examiner!

j. Check back as you go along for accuracy. If you can catch those silly little mistakes at once, your job is easier than if you have to go over the whole thing later and perhaps find a mistake near the start. However, if you do have extra time at the end, make the most of it to recheck and make sure you have done your best.

IV

SPECIAL HELPS FOR STUDYING PARTICULAR AREAS OF MATHEMATICS

17. Algebra

a. How to distinguish an equation from an expression

You will find that a great deal of your algebra can be divided into the study of equations on the one hand and the study of algebraic expressions on the other. Your whole understanding of algebra will be increased if you learn to recognize each of them quickly. It may help to compare the two as follows:

Algebraic expression	**Algebraic equation**
An *expression* is a symbol for a number. It may consist of several mathematical symbols. e.g.: $3x + 5x$	An *equation* is a statement that two expressions are symbols for the same number. e.g.: $3x + 5x = 16$
We *simplify* or *transform* expressions. e.g.: $3x + 5x = 8x$	We *solve* equations. e.g.: $3x + 5x = 16$ $8x = 16$ $x = 2$
An *identity* is an equation which is true for *all* values of the variables for which the expressions are defined. e.g.: $3x + 5x = 8x$	*A conditional equation* is an equation which is true for *not all* values of the variables for which the expressions are defined. e.g.: $3x + 5x = 16$
An identity exhibits two expressions for the *same* number.	The two expressions represent *different* numbers except for

certain values of the variables. We are interested in finding those numbers for which the equation is true. We call the numbers *roots* or *solutions* of the equation.

The operations which can be performed on an *expression* (one member of an identity) are those which *will not change the value* of the expression. These include:

(1) substituting a quantity for its equal
(2) performing indicated operations
(3) multiplying or dividing by one (e.g., in the form $\frac{a}{a}$)
(4) adding zero (e.g., in the form $a - a$)
(5) any other algebraic operation such as factoring, which is known not to change the value of the expression.

The operations which can be performed on an *equation* are those which *will not change the roots of the equation*. These include:

(1) performing the same operation with the same number (with certain limitations) on both members of the equation (e.g., adding 17 to both members, taking square roots of both members, taking logarithms of both members)
(2) performing on one member of the equation any of the operations which is permissible on an expression (see left column).

b. How to solve story problems

If you are one of those students who think story problems are hard, the chances are that you haven't yet learned how to read carefully and to organize data meaningfully. In the following plan of suggested procedure, steps 1, 2, and 3 are the parts where you organize your data.

(1) *Read*, don't just skim over the words. Get a clear picture in your mind of the whole situation. Draw a sketch of it if possible. Look up any words you don't understand.

(2) *Indicate the meaning of the unknown.* Decide what quantity (or quantities) you are going to choose for your unknown (or unknowns). Usually your best choice is a

quantity you are trying to find. Give it a letter and write down clearly this important part of the data, *showing the unit of measure*. Suppose a problem asks: "How much water must be added to 20 ounces of a solution of alcohol and water which is 80 per cent alcohol to produce a solution which is 60 per cent alcohol?" You would write, "Let n = the number of ounces of water added to the original solution," *not* "Let n = the water." Habitually starting with the words "the number of" helps to make your statement clear.

(3) *Represent other quantities in terms of the unknown.* This part of organizing data is often neglected. For clear understanding you must write down in terms of the unknown any other quantities with which you may need to deal in order to solve the problem. For example, if n is specified as above, write:

"Then $20 + n$ = no. of oz. of final solution
$.80\ (20)$ = no. of oz. of alcohol in original solution
$.60\ (20 + n)$ = no. of oz. of alcohol in final solution"

(4) *Write the equation.* If the preceding steps have been done carefully, writing an equation will be comparatively easy. You need simply to find two different ways to express one of the quantities in the problem and equate them. Since the amount of pure alcohol has not changed from the original solution to the final solution, the equation for the above problem is:

"$.80\ (20) = .60\ (20 + n)$."

(5) *Solve the equation.* Use sound algebraic methods.

(6) *Give the answers.* Reread the problem to be sure that you answer the questions asked. Label your answers, showing in what units they are measured; e.g., "6⅔ *ounces* of water added."

(7) *Check.* Verify whether the answers you have obtained satisfy all the requirements of the words in the original story problem. Substituting in the equation you made is *not* a valid check of the problem because your error may have been in formulating your equation.

Note that out of the seven steps above, six involve reading or rereading the problem.

c. How to develop accuracy in using logarithms

(1) Do your work neatly.

(2) Work systematically. When you have a logarithmic computation to do, first make a complete plan of your work, leaving spaces for the numbers you will use and showing clearly what operations to perform on them. Do this before you touch the tables. For instance to compute

$$\frac{23.68 \times 143.6^2}{983}$$

a possible plan would be:

$$\begin{aligned} \log 23.68 &= \\ + \log 143.6^2 = 2(\quad) &= \\ \log \text{numerator} &= \\ - \log 983 &= \\ \log \text{answer} &= \\ \text{answer} &= \end{aligned}$$

The wavy line is a convenient device to remind you to subtract at that point.

Those of you who use cologs could write:

$$\begin{aligned} \log 23.68 &= \\ \log 143.6^2 = 2(\quad) &= \\ \operatorname{colog} 983 &= \\ \log \text{answer} &= \\ \text{answer} &= \end{aligned}$$

(3) When you are interpolating in the table, check that the number you find is reasonable: is it between the right two numbers in the table; is it nearer to the right one of these?

(4) Make an estimate from your original expression of approximately what the final answer should be. If your com-

puted answer is far from your estimate, look for errors, especially in your characteristics—or perhaps in your original plan.

(5) The habit of continuous checking back, mentioned in Article 9, can help a great deal in work with logarithms.

18. Geometry

a. How to study theorems

The main thing to remember about studying theorems is that it is a matter of understanding rather than memorizing. If you find yourself learning by rote and not grasping the meaning of the theorem, you should realize that you are wasting your time. Start working for understanding. Get some help from the teacher if necessary to get you started right.

For studying a theorem which is written out in the book, the following suggestions may help.

(1) Have a "study sheet" ready.

(2) Read the statement of the theorem and study the drawing for clear understanding of the "Given" and "To prove."

(3) Make your own drawing on your "study sheet."

(4) Close the book without reading further, and try proving the theorem yourself as you would an original exercise. If you are successful, check your proof against the one in the book. (Of course it is possible that two different proofs may be equally correct. If you think you have discovered a new proof, discuss the matter with your teacher.)

(5) If you can't prove it yourself, take a quick look at the proof in the book, and then try again to write a proof with the book closed.

(6) If you still can't prove it, carefully work through the explanation in the book carrying out details on your "study sheet" as you go along. Find out how it *is* proved.

(7) Finally, before you leave your study of this theorem, whether you did it yourself or had to have help, analyze

the proof sufficiently to be able to state the main plan of the proof.

b. How to organize the definitions, axioms, postulates, and theorems of geometry into a body of knowledge which you understand

As you do your proofs in geometry, you constantly need to give reasons or authorities for the statements in your proofs. Certain definitions, axioms, postulates, and theorems, which have been agreed upon or proved earlier, are used for these authorities. You probably realize by now that a good mastery of the theorems, etc., is very necessary for the deductive reasoning used in geometry. How can you make sure that these authorities are ready to serve you when you need them?

For many of the theorems, the frequent use you make of them fixes them in your mind automatically, but there are other theorems which you do not use often enough to master in this way. Probably the two things that help most for mastery are: (1) associating the ideas of the theorems, etc., with the geometric picture involved so that you have a quick connection of picture, idea, and statement of the theorem, in that order, and (2) being aware of the grouping of the theorems according to the kind of thing they can be used to prove, so that when you need to prove, say, that lines are parallel, you can quickly think over all the methods you have had of proving lines parallel.

To develop your picture-theorem association, a periodic systematic review, perhaps once a week, is helpful. One way to do this is to leaf through the book with a card in your hand, cover with the card the statement of each theorem as you come to it, look at the drawing, and try to state the theorem it illustrates. Then reverse the process: cover the drawing, read the theorem, and see how quickly you can sketch the drawing and state the "Given" and "To prove." This method can be improved if as you go through the course you make yourself theorem cards for the new theorems you have each day. Write the statement of the theorem (not the proof) on one side of a 3″ x 5″ card and on the other side make a drawing to go with it. These cards can be shuffled and you can do the practice described above by looking at one side, trying to reproduce the other side, and checking by turning over the card. Once in a while it helps to

cover several sheets of paper with a jumble of drawings in different shapes and positions from those in the book. Then try pointing to drawings and quickly stating the theorem, the "Given" and "To prove." Try doing this with one of your classmates.

To develop your awareness of the groups of theorems, you need to see the theorems in those groups. Some books give lists of theorems grouped according to whether they can be used to prove angles equal, or to prove lines parallel, etc. Such lists can be very helpful if they are used intelligently. Intelligent use means not just looking at them once, and not turning to them every time you do a proof, but developing day by day, with the help of the printed lists, corresponding lists in your understanding. If there are no such lists in your book, or even if there are, the theorem cards mentioned above can help you make your own lists. In one corner of each card write what that theorem can be used to prove. Then once a week or so, sort the cards according to your notes in the corner, and see how the various groups of theorems grow. The practice of picture-theorem association described above can be combined with developing mastery of the groups. Some students prefer, instead of using the cards, to list the theorems by groups in a notebook as they are developed. This method is probably not as good as the card method, but it is not without value.

We have spoken here mostly of the theorems. The axioms, postulates, and definitions can be classified in much the same way along with the theorems.

c. *How to think out an original proof*

If you have a good mastery of the theorems and other authorities in geometry and you know how to classify them as described in the last section, the chances are that original proofs come rather easily to you. When you don't see quickly how to prove something, however, the most fruitful way of finding a method of proof is usually to reason backward from the conclusion. When a teacher helps you find a method of proof, he usually doesn't tell you what to do, but he asks some questions which help you discover the method. When the teacher isn't there to help, you can ask yourself some of the same questions, such as:

What am I trying to prove?

What are some possible methods of proving that?

Is this a method that is likely to work here?

If I am going to use congruent triangles, which triangles are best to use?

Do I have enough information to prove the triangles congruent?

Am I making use of all the facts in the "Given"?

Does every step have a purpose in my progress toward the final conclusion?

At every step does the reason I have given fulfill both of the following requirements:

(1) that the *hypothesis* of the reason is satisfied by particular statements in earlier steps?

(2) that the *conclusion* of the reason is in agreement with the statement I am proving in this step?

Get into the habit of asking such questions of yourself as you work and you will learn to do some good original thinking.

d. How to see the logic in geometry

You have probably noticed that much of the time in geometry you are talking about how to reason clearly. This study of how men think is one of the most valuable parts of the course. To get the most out of this phase of geometry it might help to have a section in your notebook devoted to logical thinking.

First you would want a list of some of the kinds of reasoning, such as deductive and inductive reasoning, and indirect reasoning. Write a little about each one and as you use them from day to day look back at what you have written. You may want to change or add to what you have said. You will probably want something in your notebook about the nature of proof, about "hypothesis" and "conclusion," and about different kinds of statements like axioms, theorems, converses and inverses, and perhaps contrapositives.

Actually there are relatively few concepts from logic in the geometry you study, but the ones that *are* there are very fundamental. Think about them and discuss them, not only in geometry but in your daily living.

19. *Trigonometry*

a. How to prove identities and solve equations in trigonometry

Here again is a situation for which no magic medicine will make everything easy. However, there is a great deal that you can do for yourself. As in all studying, being aware of the nature of the job you are doing, and knowing and using definite guiding principles, are basic to your success.

In proving an identity you wish to transform a given expression *without changing its value* until it is identical with a second given expression. This means you must (1) know what transformations can be made without changing the value and (2) watch the second expression to be sure any transformations you make are really making your first expression more like the second expression.

At this point you might be wise to refer back to the discussion (Article 17:a) of the operations which are permitted on expressions in identities. If you fully grasp the idea of permissible operations, you have the key to the whole subject of identities. Briefly the transformations which do not change the value of a trigonometric expression include (1) substitutions from your basic trigonometric identities (such as substituting $1 - \cos^2 x$ for $\sin^2 x$) and (2) algebraic transformations such as factoring, reducing fractions, simplifying complex fractions, etc. You see then, why it is so important (1) *to know your trigonometric formulas very well*, and (2) *to be sure of your elementary algebra*. If your algebra is rusty, get an algebra book and review the type of thing you need in trigonometry. (See Article 19:d.) Notice that a permissible operation which is often helpful is to multiply a fraction by *one* in a form such as $\frac{1 - \cos x}{1 - \cos x}$. In this way you can often supply factors which you see from the second expression are needed.

As you watch the second expression to see what changes you need to make, there are three main approaches you may be able to take. Watch for the signposts that point to:

(1) the need for making the angles alike

(2) the need for making the trigonometric functions alike
(3) the need for making the algebraic forms alike.

For instance, if one expression is a single fraction and the other is the sum of two fractions, it would be natural to make their algebraic forms alike by performing the addition of the two fractions. If you see no obvious transformation to make the functions alike, try changing all of them into sines and cosines first.

In solving trigonometric equations, just as in solving conditional equations in algebra, you have greater freedom in performing operations than you do in proving identities. If you refer back to Article 17:a, you will recall that not only may you perform on one member of the equation any of the operations permitted in proving an identity, but also you may operate on both sides of the equation at once according to the axioms of equality. The important thing here is to keep each equation *equivalent* to the preceding one, i.e., having the same roots. It will sometimes be necessary to make trigonometric substitutions before you can proceed to the algebraic solution of the equation. The substitutions may be such as to make the angles alike or the functions alike, or occasionally to set up a product rather than a sum. Note that you will frequently use here the principle that if a product of factors is equal to zero, one of the factors must be zero. The *algebraic* solution obtained will be the value of a trigonometric *function*; $\sin \theta = \frac{1}{2}$. Remember that the unknown for which you are solving is the *angle*. Also remember that there are many angles (e.g., $\theta = 30° + n \cdot 360°$, $150° + n \cdot 360°$) for which $\sin \theta = \frac{1}{2}$.

b. How to master trigonometric formulas

Many of the more important trigonometric formulas will quickly become familiar through use. For help on mastering the less familiar ones, see the section on how to memorize, Article 5.

c. How to develop accuracy in trigonometric computations

See the section on how to develop accuracy in using logarithms, Article 17:c.

d. How to review the algebra you need in trigonometry

In order to prove identities and solve trigonometric equations, some facility with algebra is needed. You may find you have

forgotten a good deal of your algebra. With a little effort you can review early in your trigonometry course the algebra you need, and thus make your trigonometry easier from the start. If you still have one of your algebra books, it will serve the purpose. Otherwise try to borrow one. The most important topics you will need to review are:

(1) Factoring (only the simpler types)
(2) Solving equations by factoring
(3) Fractions: the fundamental principle of fractions, reducing fractions, adding fractions, simplifying complex fractions
(4) Fractional equations
(5) Solving quadratic equations by the quadratic formula
(6) Radicals and radical equations

Go through these sections in the book with a "study sheet" and pick out the main ideas that you need to recall. Then do some practice on each type, using exercises from the book to redevelop your skill and understanding. Keep your list of ideas on the "study sheet" to refer to again if you continue to have trouble.

If you have trouble seeing the connection between the trigonometric expressions and the corresponding algebraic ideas, try substituting a letter for a trigonometric function temporarily. For instance if you don't see that $4 \sin^2\theta - 5 \sin \theta + 1$ is factorable, let k represent $\sin \theta$ temporarily and factor $4k^2 - 5k + 1$; then replace k by $\sin \theta$. Try, however, to develop the ability to think of the function as though it were a single letter and not depend on the actual substitution of a letter.

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